First-Order Logic

Part One

Recap from Last Time

Recap So Far

- A ***propositional variable*** is a variable that is either true or false.
- The ***propositional connectives*** are as follows:
  - Negation ("not"): $\neg p$
  - Conjunction ("and"): $p \wedge q$
  - Disjunction ("or"): $p \vee q$
  - Truth ("true"): $\top$
  - Falsity ("false"): $\bot$
  - Implication ("implies"): $p \to q$
  - Biconditional ("if and only if"): $p \leftrightarrow q$

p	q	$p \to q$
F	F	___
F	T	___
T	F	___
T	T	___

What's the truth table for the → connective?

What's the negation of $p \to q$?

Operator Precedence

- How do we parse this statement?

$$\neg x \to y \lor z \to x \lor y \land z$$

- Operator precedence for propositional logic:

$$\neg$$

$$\land$$

$$\lor$$

$$\to$$

$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- The main points to remember:
  - $\neg$ binds to whatever immediately follows it.
  - $\wedge$ and $\vee$ bind more tightly than $\rightarrow$.
- We will commonly write expressions like $p \wedge q \rightarrow r$ without adding parentheses.
- For more complex expressions, we'll try to add parentheses.
- Confused? ***Please ask!***

New Stuff!

First-Order Logic

What is First-Order Logic?

- ***First-order logic*** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
  - ***predicates*** that describe properties of objects,
  - ***functions*** that map objects to one another, and
  - ***quantifiers*** that allow us to reason about multiple objects.

Some Examples

$$Likes(You, Eggs) \land Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

$$Learns(You, History) \lor ForeverRepeats(You, History)$$

$$In(MyHeart, Havana) \land TookBackTo(Him, Me, EastAtlanta)$$

$$Likes(You, Eggs) \land Likes(You, Tomato) \to Likes(You, Shakshuka)$$

$$Learns(You, History) \lor ForeverRepeats(You, History)$$

$$In(MyHeart, Havana) \land TookBackTo(Him, Me, EastAtlanta)$$

These blue terms are called constant symbols. Unlike propositional variables, they refer to objects, not propositions.

$$Likes(You, Eggs) \land Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

$$Learns(You, History) \lor ForeverRepeats(You, History)$$

$$In(MyHeart, Havana) \land TookBackTo(Him, Me, EastAtlanta)$$

The red things that look like function calls are called *predicates*. Predicates take objects as arguments and evaluate to true or false.

$$Likes(You, Eggs) \land Likes(You, Tomato) \to Likes(You, Shakshuka)$$

$$Learns(You, History) \lor ForeverRepeats(You, History)$$

$$In(MyHeart, Havana) \land TookBackTo(Him, Me, EastAtlanta)$$

What remains are traditional propositional connectives. Because each predicate evaluates to true or false, we can connect the truth values of predicates using normal propositional connectives.

Reasoning about Objects

- To reason about objects, first-order logic uses ***predicates***.
- Examples:

$$\mathit{Cute}(\mathit{Quokka})$$

$$\mathit{ArgueIncessantly}(\mathit{Democrats}, \mathit{Republicans})$$

- Applying a predicate to arguments produces a proposition, which is either true or false.
- Typically, when you're working in FOL, you'll have a list of predicates, what they stand for, and how many arguments they take. It'll be given separately than the formulas you write.

First-Order Formulas

- Formulas in first-order logic can be constructed from predicates applied to objects:

$$Cute(a) \to Dikdik(a) \lor Kitty(a) \lor Puppy(a)$$

$$Succeeds(You) \leftrightarrow Practices(You)$$

$$x < 8 \to x < 137$$

The less-than sign is just another predicate. Binary predicates are sometimes written in *infix notation* this

Numbers are not "built in" to first-order logic. They're constant symbols just like "You" and "a" above.

Equality

- First-order logic is equipped with a special predicate **=** that says whether two objects are equal to one another.
- Equality is a part of first-order logic, just as → and ¬ are.
- Examples:

$$MilesMorales = SpiderMan$$

$$MorningStar = EveningStar$$

- Equality can only be applied to ***objects***; to state that two ***propositions*** are equal, use ↔.

Let's see some more examples.

$$\mathit{FavoriteMovieOf}(\mathit{You}) \neq \mathit{FavoriteMovieOf}(\mathit{Date}) \land \mathit{StarOf}(\mathit{FavoriteMovieOf}(\mathit{You})) = \mathit{StarOf}(\mathit{FavoriteMovieOf}(\mathit{Date}))$$

These purple terms are *functions*. Functions take objects as input and produce objects as output.

$$FavoriteMovieOf(You) \neq FavoriteMovieOf(Date) \land$$
$$StarOf(FavoriteMovieOf(You)) = StarOf(FavoriteMovieOf(Date))$$

Functions

- First-order logic allows ***functions*** that return objects associated with other objects.
- Examples:

$$\mathit{ColorOf}(\mathit{Money})$$

$$\mathit{MedianOf}(x, y, z)$$

$$x + y$$

- As with predicates, functions can take in any number of arguments, but always return a single value.
- Functions evaluate to ***objects***, not ***propositions***.

Objects and Propositions

- When working in first-order logic, be careful to keep objects (actual things) and propositions (true or false) separate.
- You cannot apply connectives to objects:

⚠ $Venus \to TheSun$ ⚠

- You cannot apply functions to propositions:

⚠ $StarOf(IsRed(Sun) \land IsGreen(Mars))$ ⚠

- Ever get confused? *Just ask!*

The Type-Checking Table

	… operate on …	… and produce
Connectives (↔, ∧, etc.) …	propositions	a proposition
Predicates (=, etc.) …	objects	a proposition
Functions …	objects	an object

One last (and major) change

Some spider is radioactive.

$$\exists s.\ (Spider(s) \land Radioactive(s))$$

∃ is the ***existential quantifier*** and says "for some choice of s, the following is true."

The Existential Quantifier

- A statement of the form

$$\exists x.\ \textit{some-formula}$$

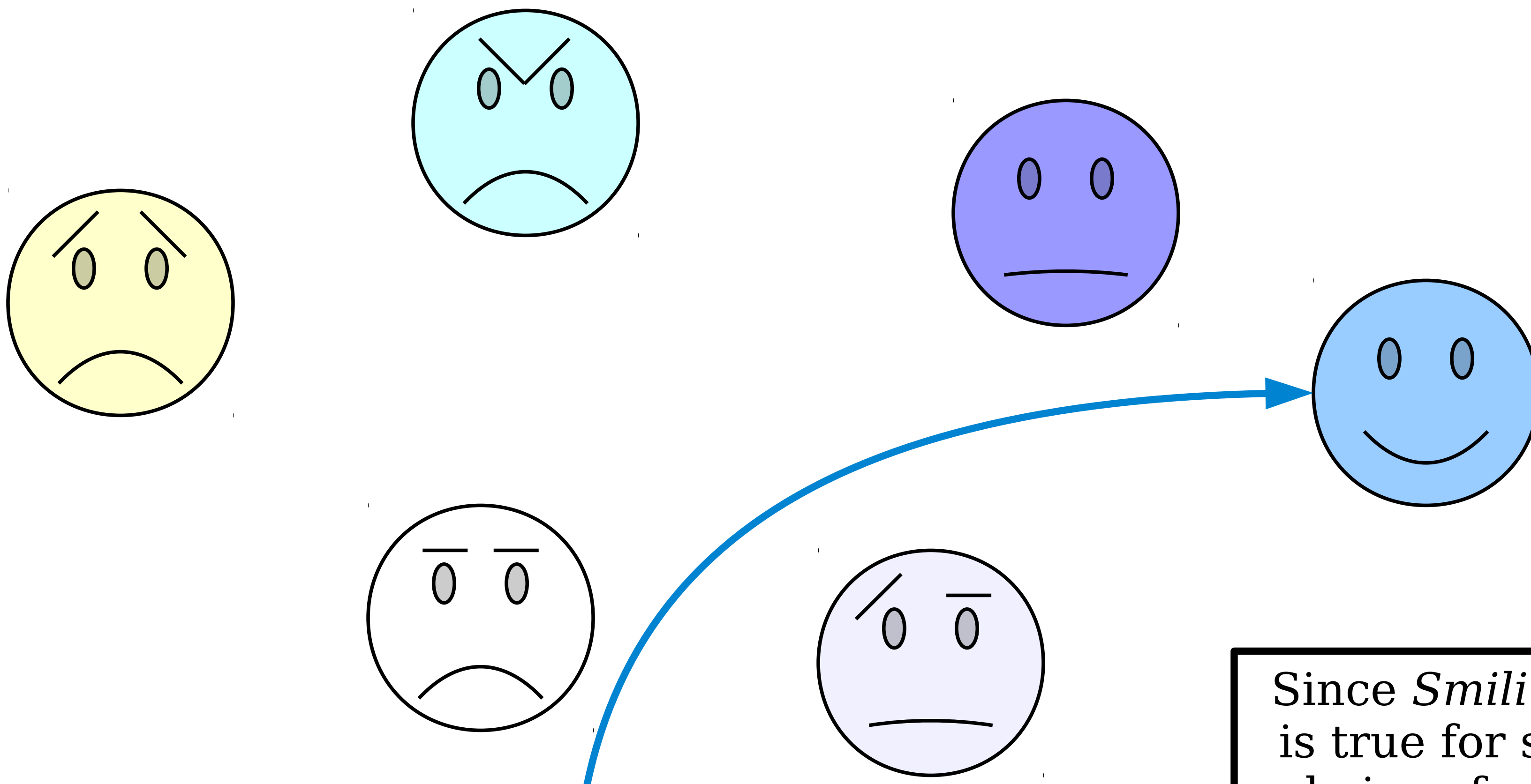
  is true if there exists a choice of x where ***some-formula*** is true when that x is plugged into it.
- Examples:

$$\exists x.\ (Even(x) \land Prime(x))$$

$$\exists x.\ (TallerThan(x, me) \land WeighsLessThan(x, me))$$

$$(\exists w.\ Will(w)) \to (\exists x.\ Way(x))$$

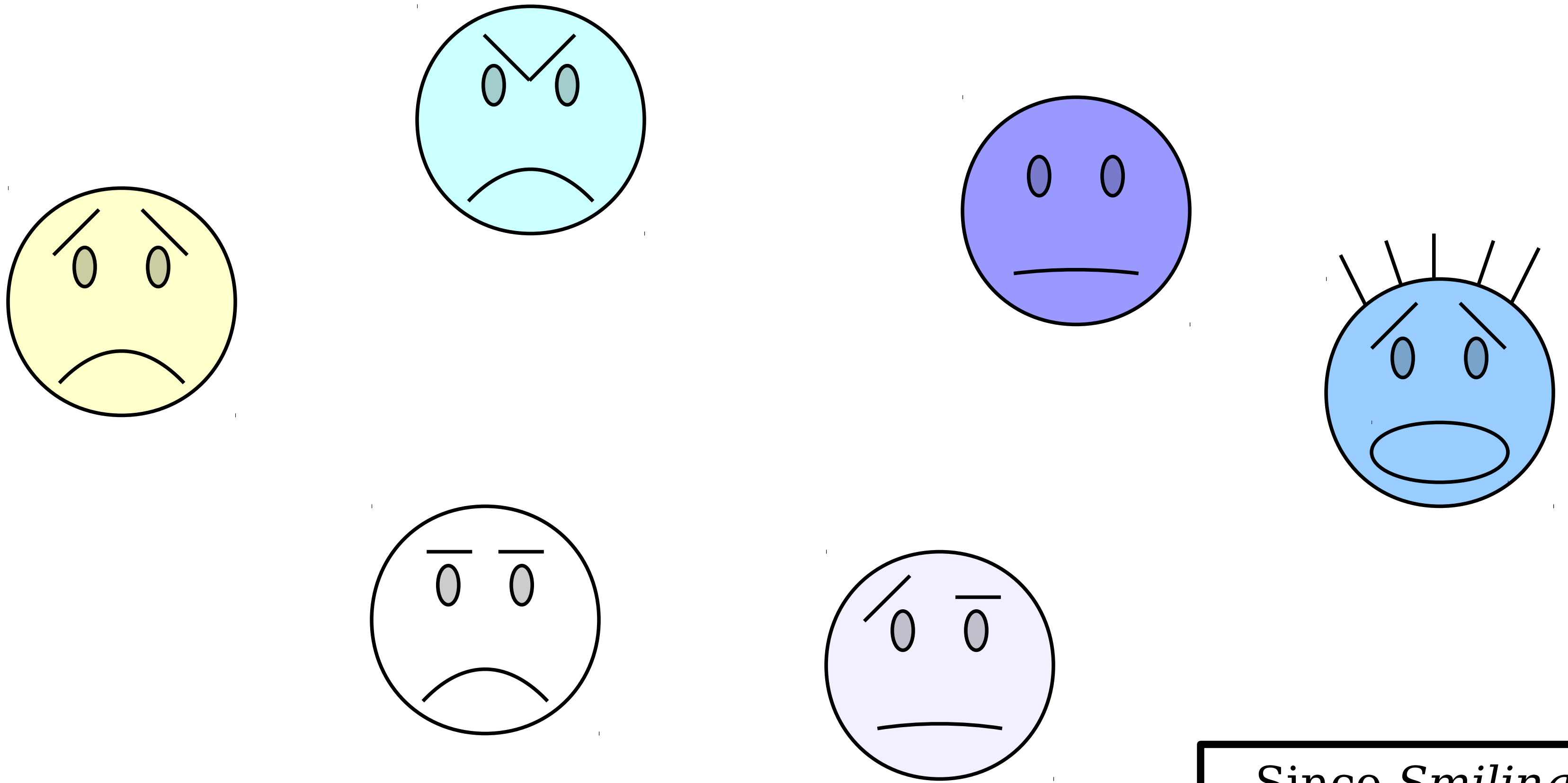
The Existential Quantifier



$\exists x.\ \mathit{Smiling}(x)$

Since *Smiling*(*x*) is true for some choice of *x*, this statement evaluates to true.

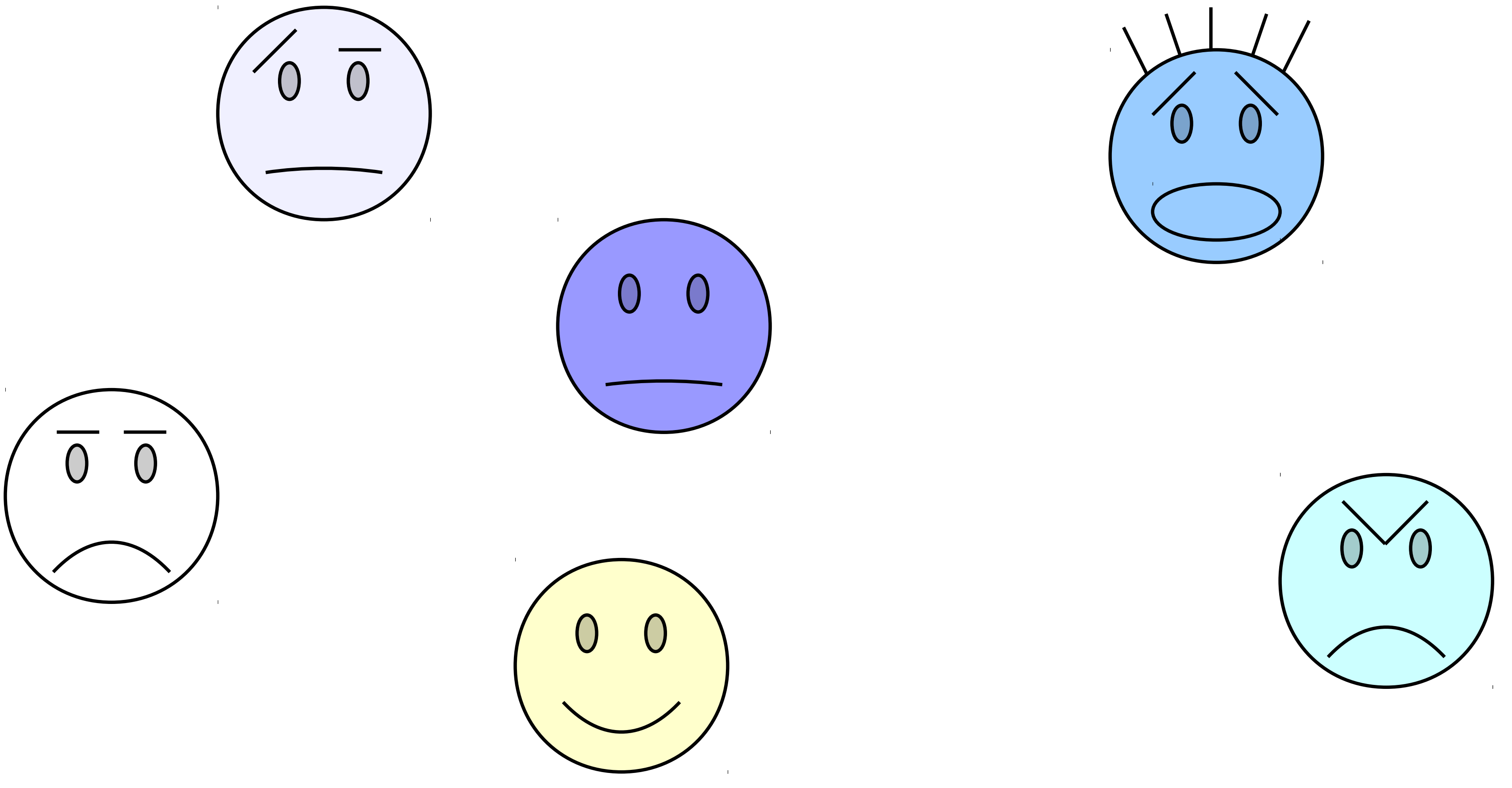
The Existential Quantifier



~~$\exists x.\ Smiling(x)$~~

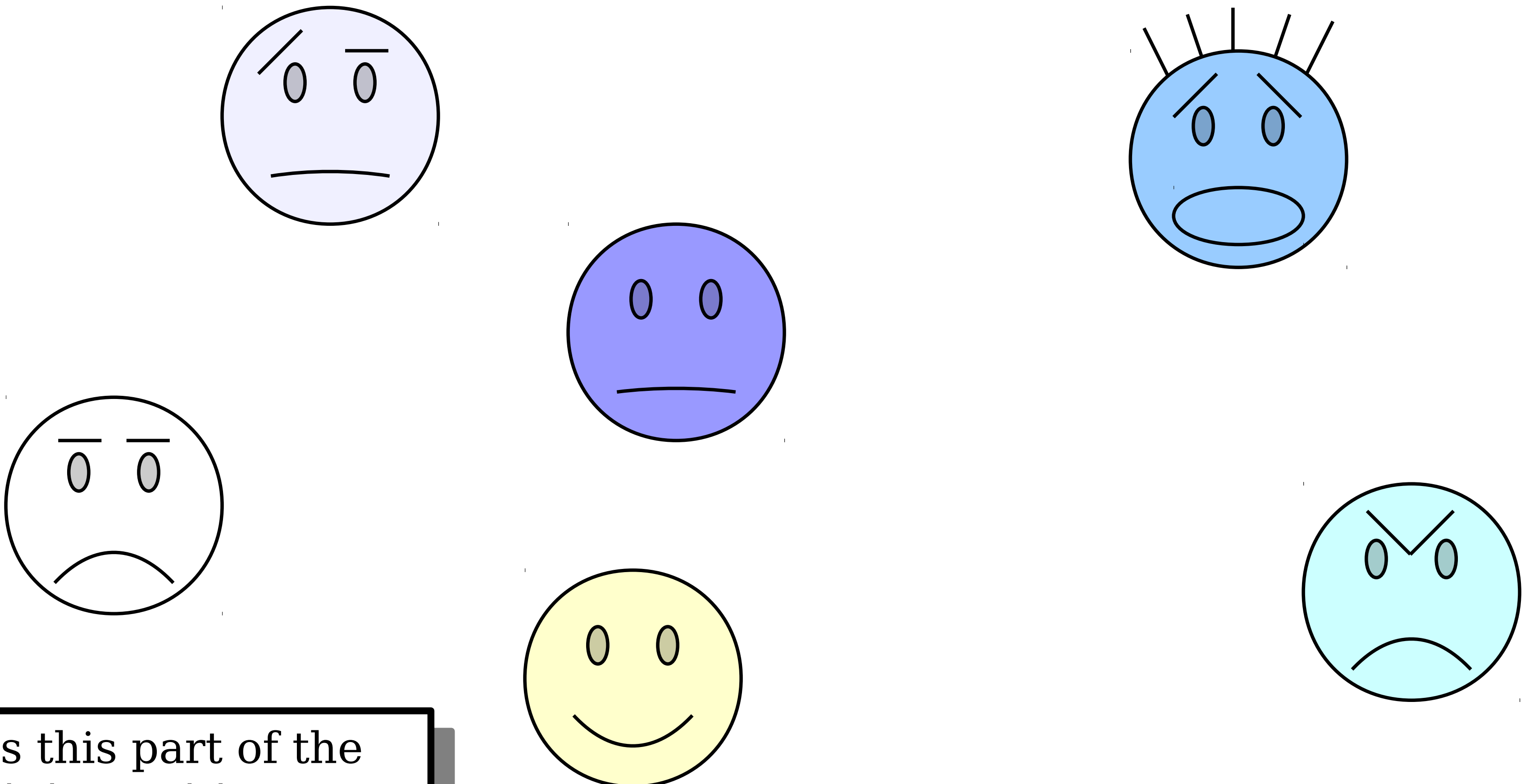
Since $Smiling(x)$ is not true for any choice of x, this statement evaluates to false.

The Existential Quantifier



$$(\exists x.\ Smiling(x)) \to (\exists y.\ WearingHat(y))$$

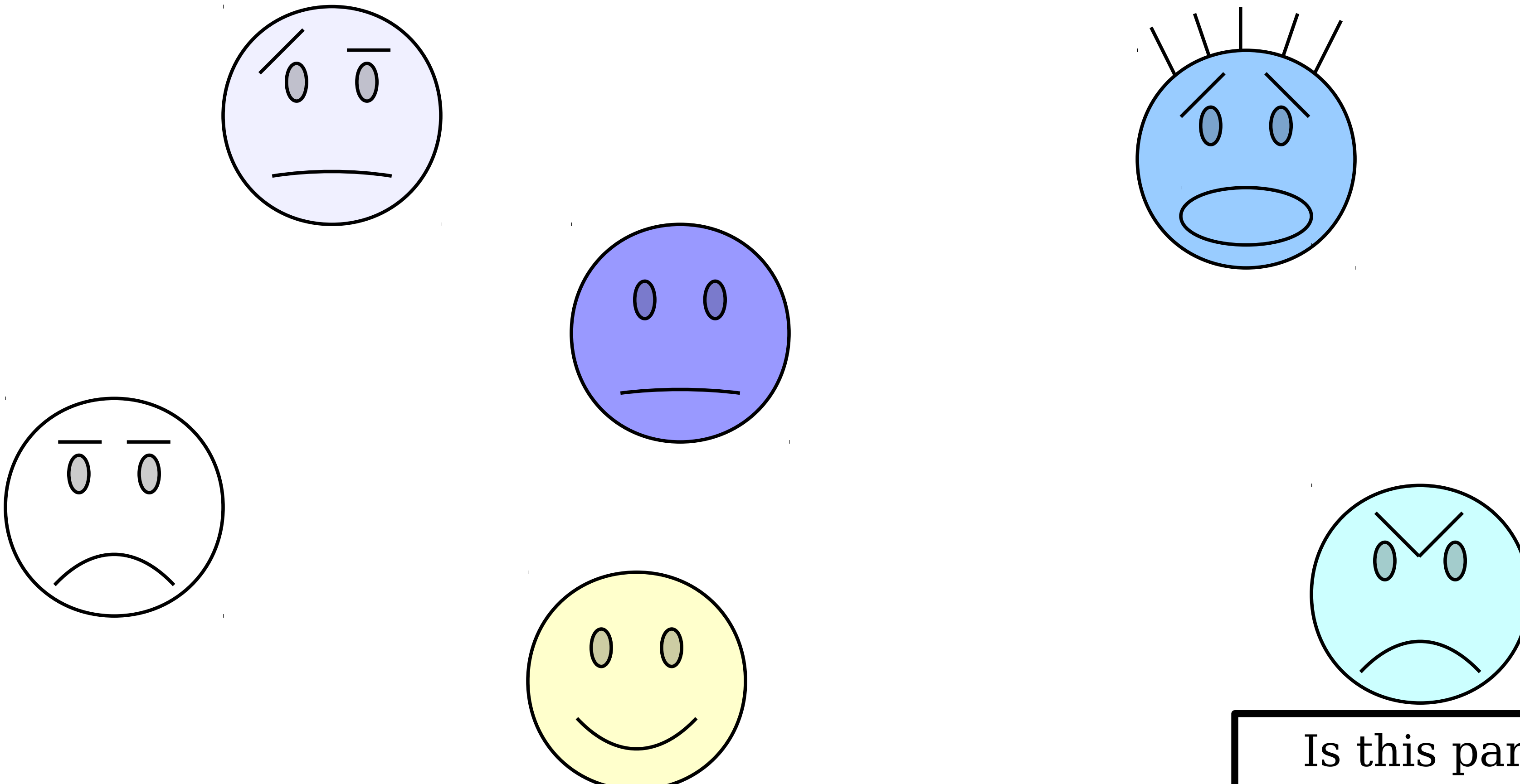
The Existential Quantifier



Is this part of the statement true or false?

$$(\exists x.\ Smiling(x)) \rightarrow (\exists y.\ WearingHat(y))$$

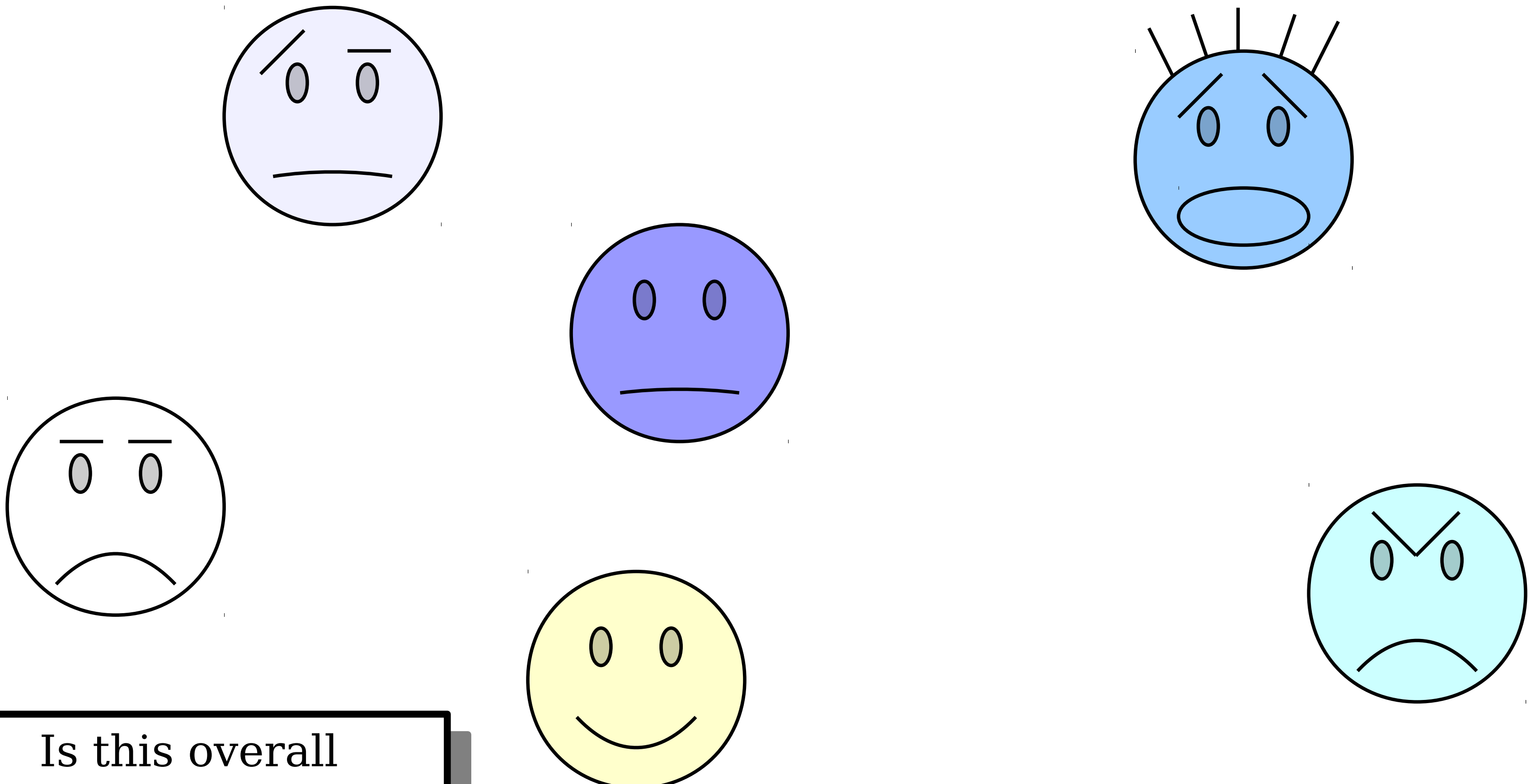
The Existential Quantifier



Is this part of the statement true or false?

$(\exists x.\ \mathit{Smiling}(x)) \rightarrow$ ~~$(\exists y.\ \mathit{WearingHat}(y))$~~

The Existential Quantifier



Is this overall statement true or false?

~~$(\exists x.\ \mathit{Smiling}(x)) \to (\exists y.\ \mathit{WearingHat}(y))$~~

Fun with Edge Cases

Existentially-quantified statements are false in an empty world, since nothing exists, period!

~~$\exists x.\ Smiling(x)$~~

Some Technical Details

Variables and Quantifiers

- Each quantifier has two parts:
  - the variable that is introduced, and
  - the statement that's being quantified.
- The variable introduced is scoped just to the statement being quantified.

$$(\exists x.\ Loves(You, x)) \land (\exists y.\ Loves(y, You))$$

The variable x just lives here.

The variable y just lives here.

Variables and Quantifiers

- Each quantifier has two parts:
  - the variable that is introduced, and
  - the statement that's being quantified.
- The variable introduced is scoped just to the statement being quantified.

$$(\exists x.\ \mathit{Loves}(\mathit{You}, x)) \land (\exists x.\ \mathit{Loves}(x, \mathit{You}))$$

The variable x just lives here.

A different variable, also named x, just lives here.

Operator Precedence (Again)

- When writing out a formula in first-order logic, quantifiers have precedence just below $\neg$.
- The statement

$$\exists x.\ P(x) \land R(x) \land Q(x)$$

  is parsed like this:

$$⚠ \quad (\exists x.\ P(x)) \ \land\ (R(x) \land Q(x)) \quad ⚠$$

- This is syntactically invalid because the variable x is out of scope in the back half of the formula.
- To ensure that x is properly quantified, explicitly put parentheses around the region you want to quantify:

$$\exists x.\ (P(x) \land R(x) \land Q(x))$$

Time-Out for Announcements!

Problem Set One

- Problem Set One was due today at 4:00PM.
  - You can extend the deadline to 4:00PM Saturday using one of your late days. As usual, no late submissions will be accepted beyond 4:00PM Saturday without prior approval.

Gradescope Tagging

- When you upload a PDF to Gradescope, please make sure to tag the pages that have your problem answers on them.
- The ***altruistic*** reason: if you don't do this, the TAs have to do it for you, and across 200+ submissions that adds up to hours of extra work.
- The ***selfish*** reason: if you don't tag the page containing a problem, Gradescope marks it as though you didn't submit it, and the TAs might give you no points because they thought you didn't submit anything.
- You can tag pages after you submit, so if you submit and then realize you forgot to tag things you can always go back and fix it.

Problem Set Two

- Problem Set Two goes out today. It’s due next Friday at 4:00PM.
  - Explore first-order logic!
  - Expand your proofwriting toolkit!
- We have some online readings for this problem set.
  - Check out the ***Guide to Logic Translations*** for more on how to convert from English to FOL.
  - Check out the ***Guide to Negations*** for information about how to negate formulas.
  - Check out the ***First-Order Translation Checklist*** for details on how to check your work.

Problem Set Two

- A few of the questions on PS2 require topics we haven’t covered yet. They’re explicitly marked as such.
- Want to start early? The Guide to Negations and Guide to First-Order Translations cover everything you need.

Back to CS103!

"For any natural number n,
n is even if and only if n^2 is even"

$$\forall n.\ (n \in \mathbb{N} \to (Even(n) \leftrightarrow Even(n^2)))$$

∀ is the ***universal quantifier*** and says "for any choice of n, the following is true."

The Universal Quantifier

- A statement of the form

  $$\forall x.\ \textit{some-formula}$$

  is true if, for every choice of x, the statement ***some-formula*** is true when x is plugged into it.

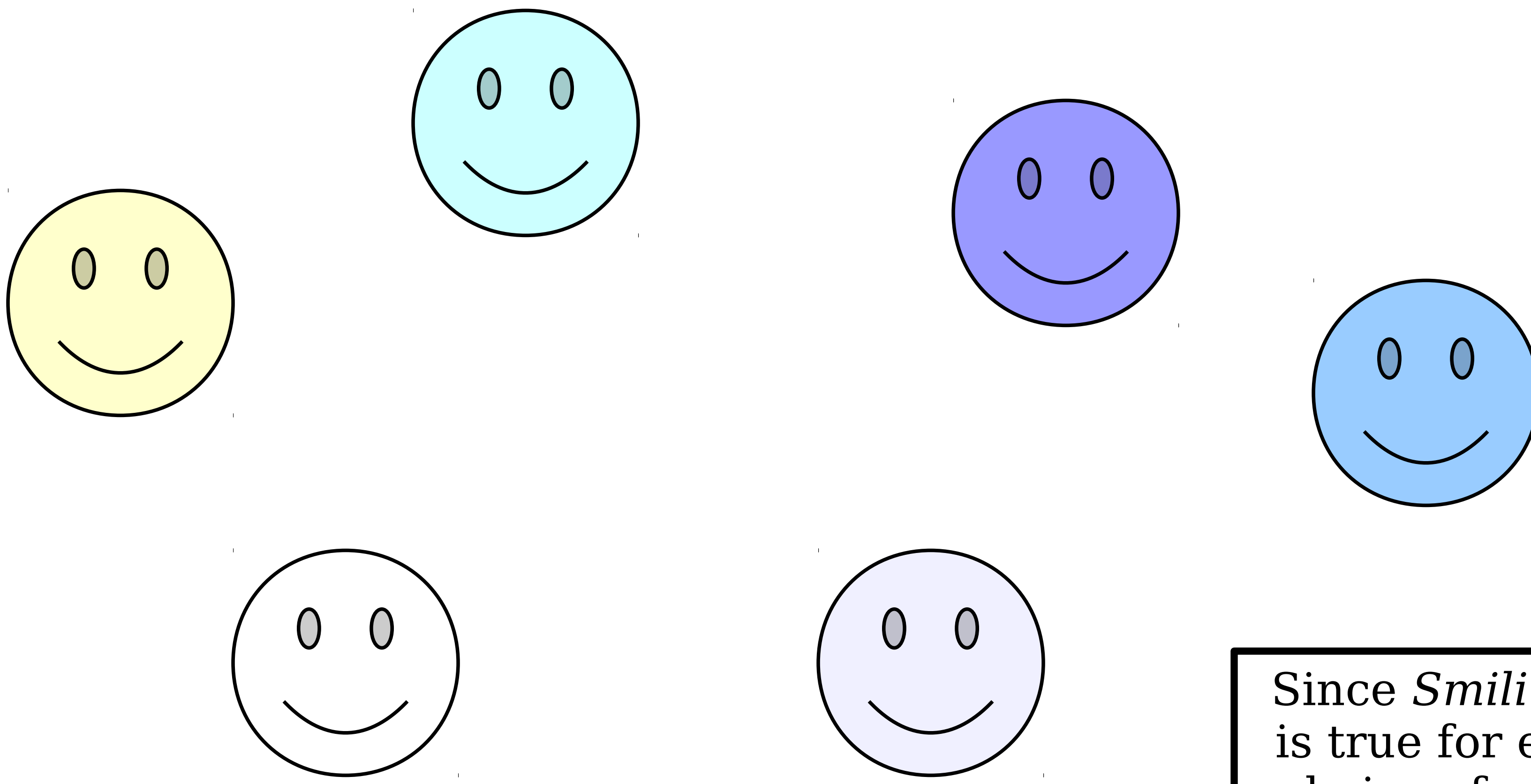
- Examples:

  $\forall p.\ (Puppy(p) \to Cute(p))$

  $\forall a.\ (EatsPlants(a) \lor EatsAnimals(a))$

  $Tallest(SultanKösen) \to$
  $\quad \forall x.\ (SultanKösen \neq x \to ShorterThan(x, SultanKösen))$

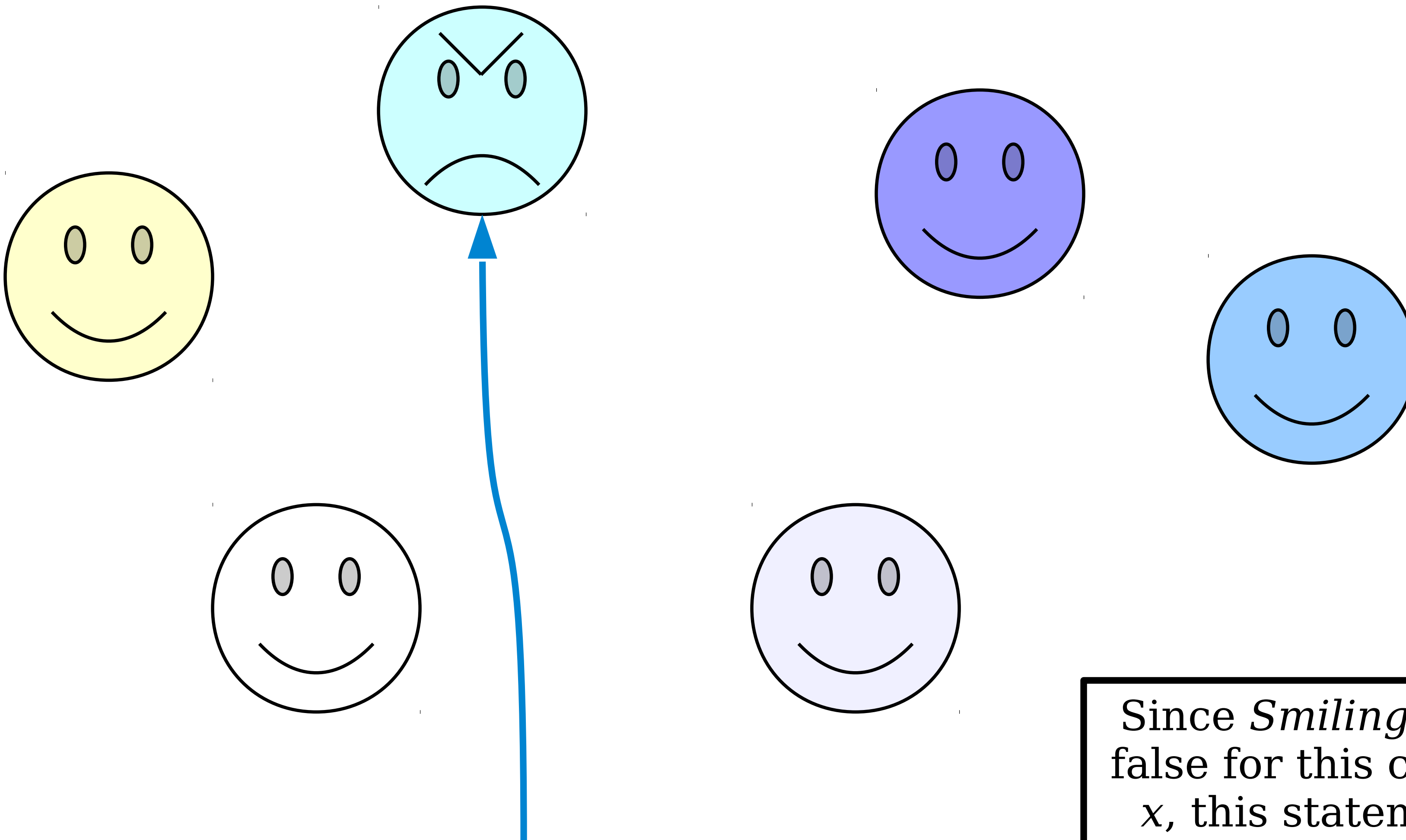
The Universal Quantifier



Since *Smiling*(x) is true for every choice of x, this statement evaluates to true.

$\forall x.\ Smiling(x)$

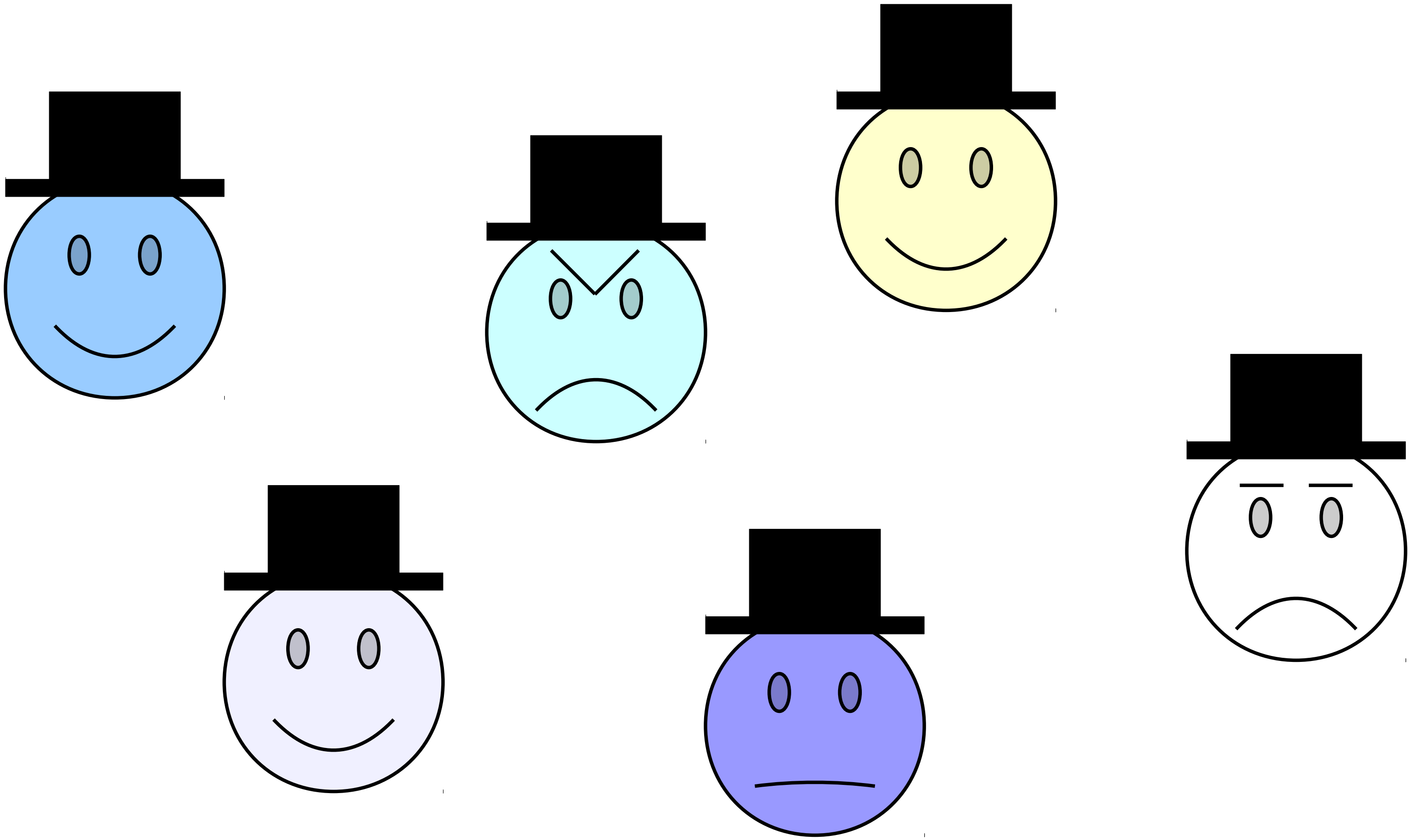
The Universal Quantifier



~~$\forall x.\ Smiling(x)$~~

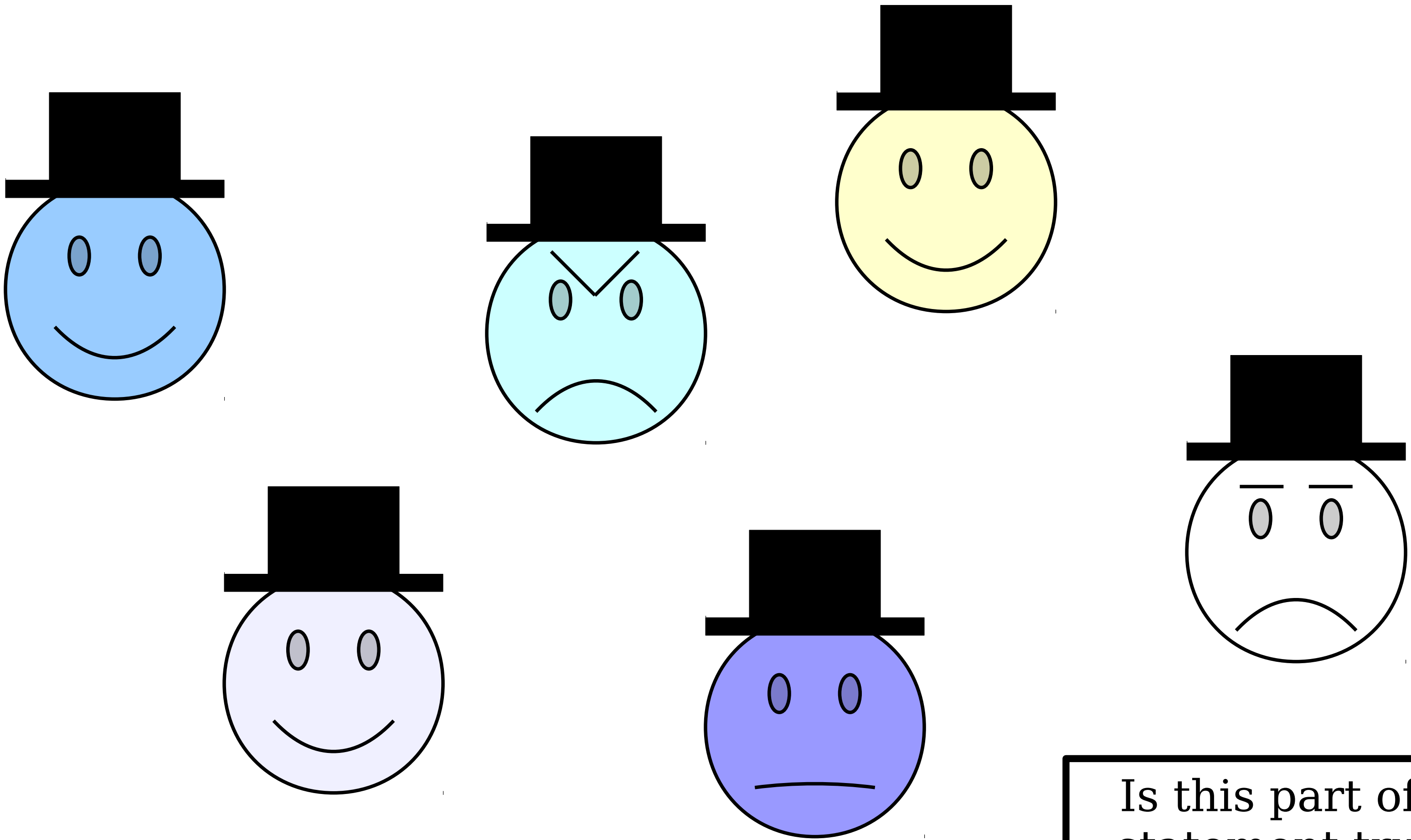
Since *Smiling*(x) is false for this choice x, this statement evaluates to false.

The Universal Quantifier



$$(\forall x.\ Smiling(x)) \to (\forall y.\ WearingHat(y))$$

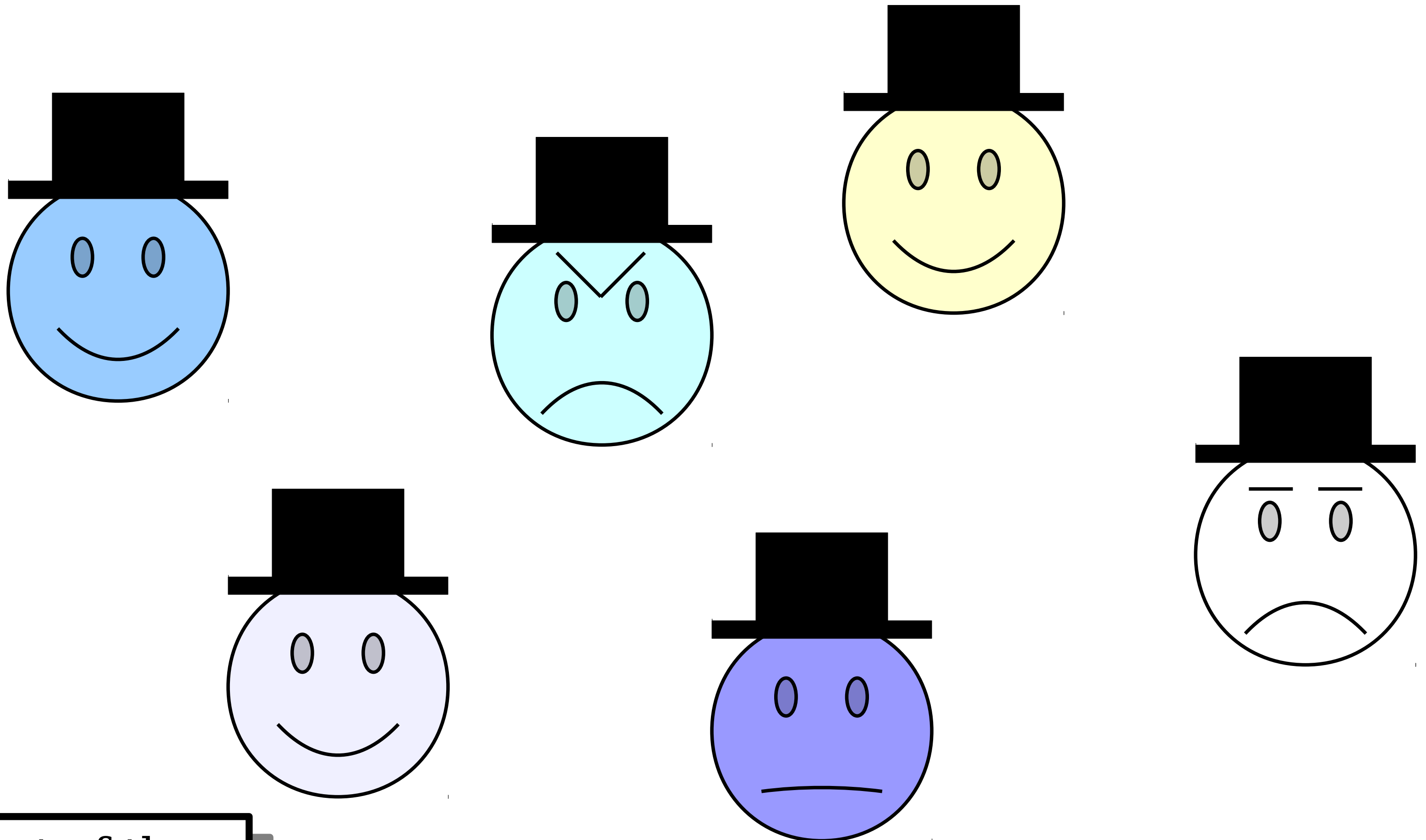
The Universal Quantifier



Is this part of the statement true or false?

$(\forall x.\ Smiling(x)) \to (\forall y.\ WearingHat(y))$

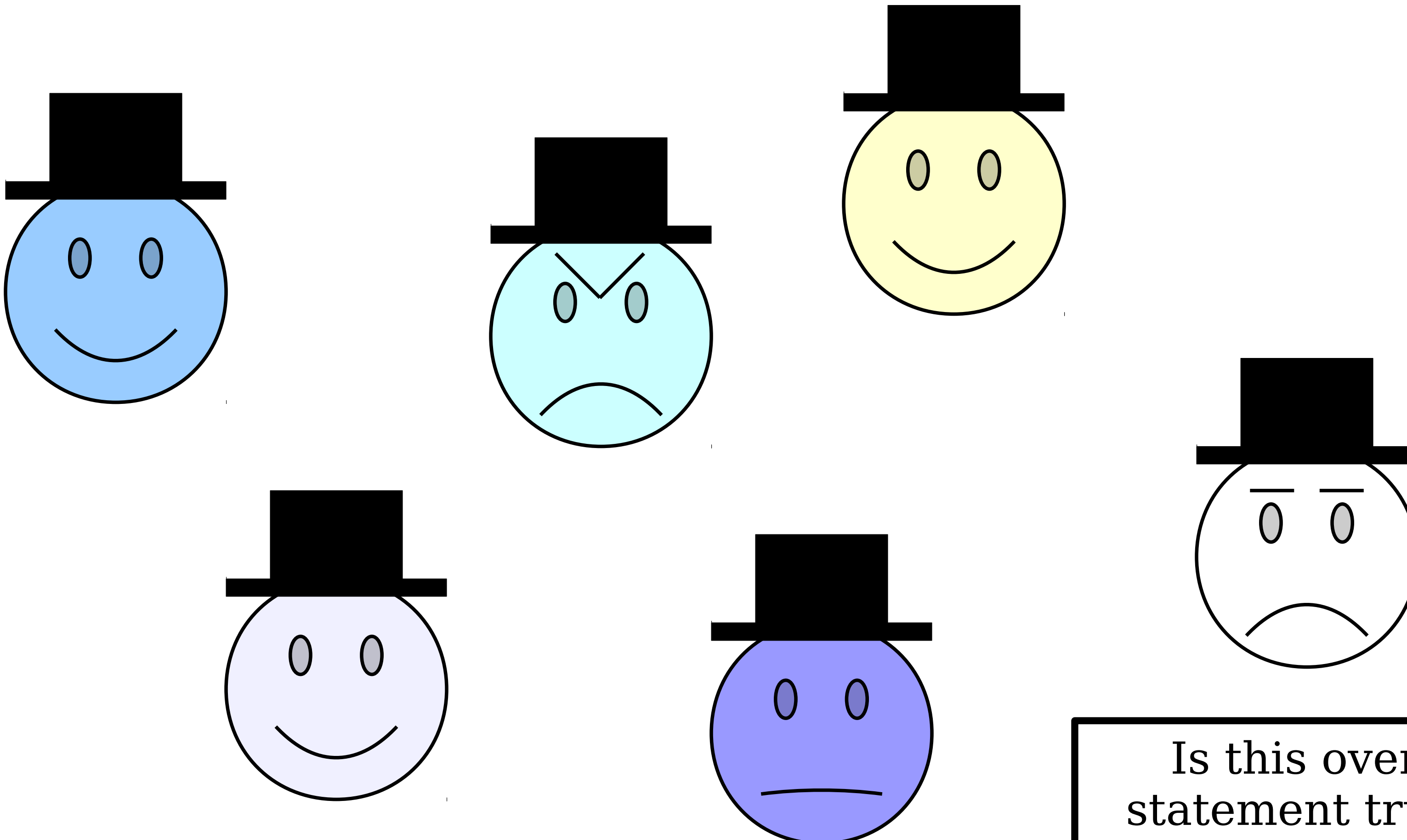
The Universal Quantifier



Is this part of the statement true or false?

~~$(\forall x.\ Smiling(x))$~~ $\rightarrow (\forall y.\ WearingHat(y))$

The Universal Quantifier



Is this overall statement true or false in this scenario?

$(\forall x.\ Smiling(x)) \to (\forall y.\ WearingHat(y))$

Fun with Edge Cases

Universally-quantified statements are said to be ***vacuously true*** in empty worlds.

$$\forall x.\ Smiling(x)$$

Translating into First-Order Logic

Translating Into Logic

- First-order logic is an excellent tool for manipulating definitions and theorems to learn more about them.
- Need to take a negation? Translate your statement into FOL, negate it, then translate it back.
- Want to prove something by contrapositive? Translate your implication into FOL, take the contrapositive, then translate it back.

Translating Into Logic

- When translating from English into first-order logic, we recommend that you

  think of first-order logic as a mathematical programming language.

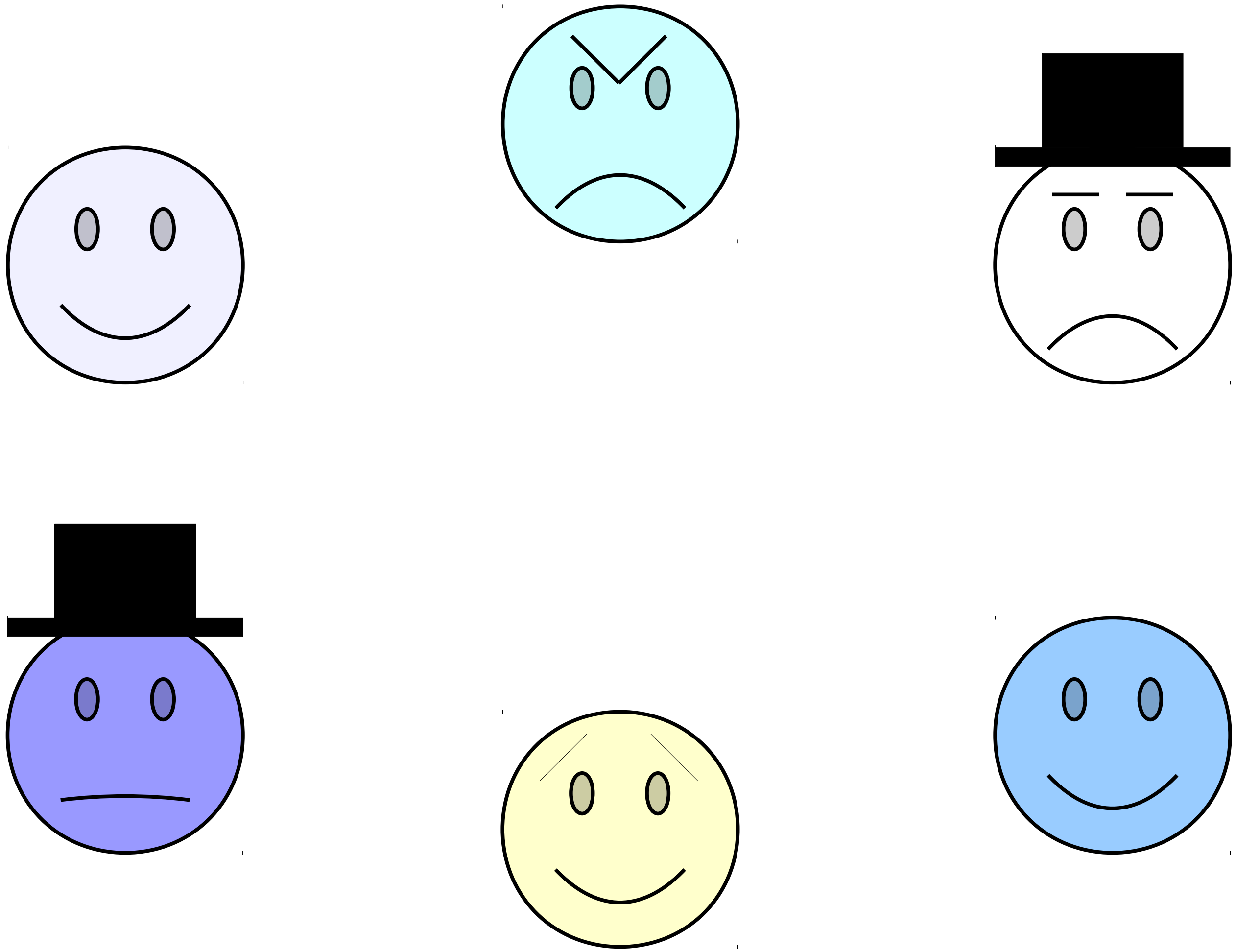
- Your goal is to learn how to combine basic concepts (quantifiers, connectives, etc.) together in ways that say what you mean.

Using the predicates

- $Smiling(x)$, which states that x is smiling, and
- $WearingHat(x)$, which states that x is wearing a hat,

write a sentence in first-order logic that says

some smiling person wears a hat.



"Some smiling person wears a hat." ***False***

$\exists x. \, (Smiling(x) \land WearingHat(x))$ ***False***

~~$\exists x. \, (Smiling(x) \to WearingHat(x))$~~ ***True***

“Some P is a Q”

translates as

$$\exists x.\ (P(x) \land Q(x))$$

Useful Intuition:

Existentially-quantified statements are false unless there's a positive example.

$$\exists x.\ (P(x) \land Q(x))$$

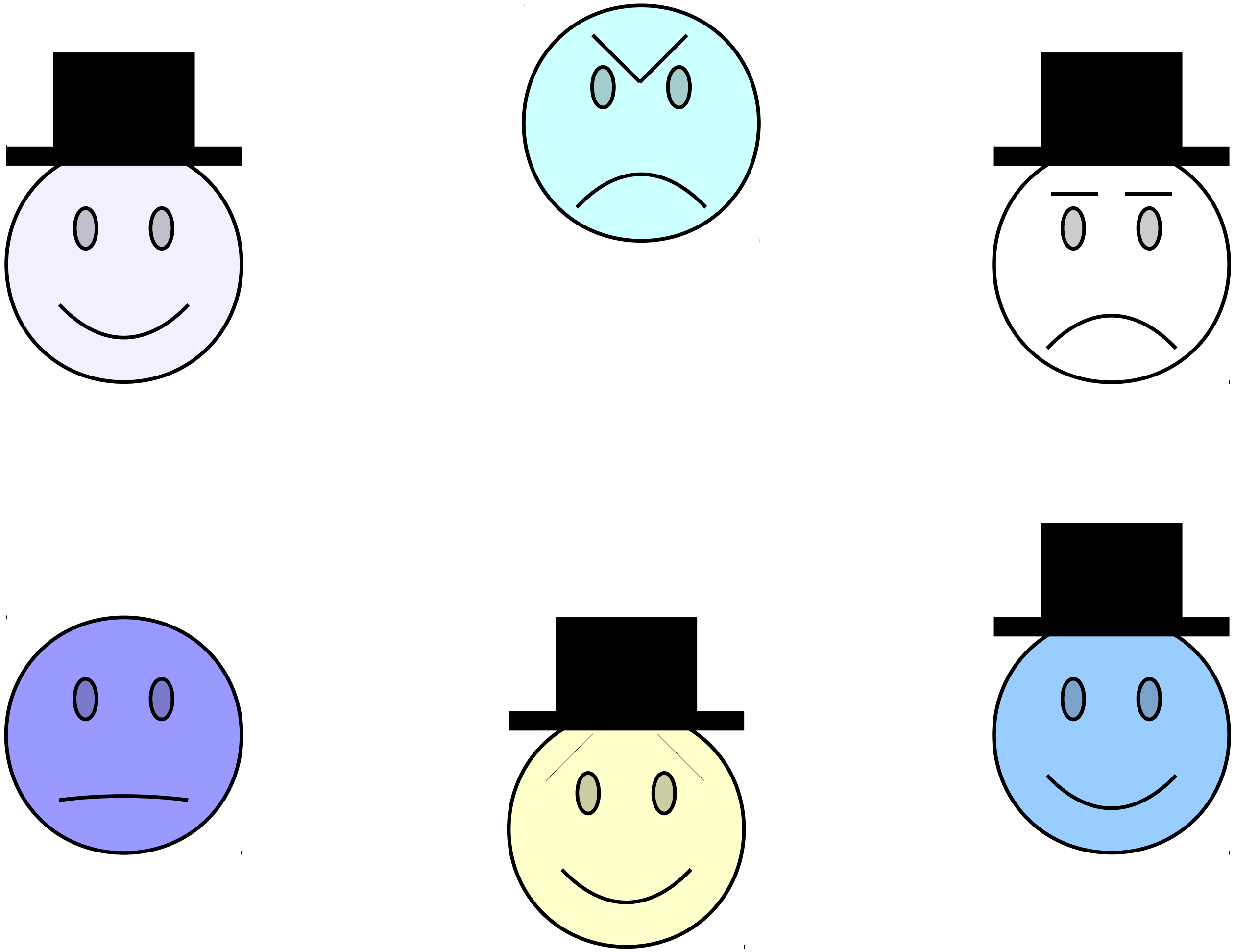
If x is an example, it must have property P on top of property Q.

Using the predicates

- $Smiling(x)$, which states that x is smiling, and
- $WearingHat(x)$, which states that x is wearing a hat,

write a sentence in first-order logic that says

every smiling person wears a hat.



"Every smiling person wears a hat."	***True***
~~$\forall x.\ (Smiling(x) \land WearingHat(x))$~~	***False***
$\forall x.\ (Smiling(x) \rightarrow WearingHat(x))$	***True***

“All *P*'s are *Q*'s”

translates as

$$\forall x.\ (P(x) \to Q(x))$$

Useful Intuition:

Universally-quantified statements are true unless there's a counterexample.

$$\forall x.\ (P(x) \to Q(x))$$

If x is a counterexample, it *must* have property P but not have property Q.

Good Pairings

- The ∀ quantifier *usually* is paired with →.

$$\forall x.\ (P(x) \to Q(x))$$

- The ∃ quantifier *usually* is paired with ∧.

$$\exists x.\ (P(x) \land Q(x))$$

- In the case of ∀, the → connective prevents the statement from being *false* when speaking about some object you don't care about.
- In the case of ∃, the ∧ connective prevents the statement from being *true* when speaking about some object you don't care about.

Next Time

- ***First-Order Translations***
  - How do we translate from English into first-order logic?
- ***Quantifier Orderings***
  - How do you select the order of quantifiers in first-order logic formulas?
- ***Negating Formulas***
  - How do you mechanically determine the negation of a first-order formula?
- ***Expressing Uniqueness***
  - How do we say there’s just one object of a certain type?